CHAPTER 10
FACTORING POLYNOMIALS

A factor of a quantity N, as defined in chapter 2 of this course, is any expression which can be divided into N without producing a remainder. Thus 2 and 3 are factors of 6, and the factors of 5x are 5 and x. Conversely, when all of the factors of N are multiplied together, the product is N. This definition is extended to include polynomials.

The factors of a polynomial are two or more expressions which, when multiplied together, give the polynomial as a product. For example, 3, x, and $x^2 - 4$ are factors of $3x^3 - 12x$, as the following equation shows:

$$(3)(x)(x^2 - 4) = 3x^3 - 12x$$

The factors 3 and x, which are common to both terms of the polynomial $3x^3 - 12x$, are called COMMON FACTORS.

The distributive principle, mentioned in chapters 3 and 9 of this course, is an important part of the concept of factoring. It may be stated as follows:

If the sum of two or more quantities is multiplied by a third quantity, the product is found by applying the multiplier to each of the original quantities separately and summing the resulting expressions. It is this principle which allows us to separate common factors from the terms of a polynomial.

Just as with numbers, an algebraic expression is a prime factor if it has no other factors except itself and 1. The factor $x^2 - 4$ is not prime, since it can be separated into $x - 2$ and $x + 2$. The factors $x - 2$ and $x + 2$ are both prime factors, since they cannot be separated into other factors.

The process of finding the factors of a polynomial is called FACTORING. An expression is said to be factored completely when it has been separated into its prime factors. The polynomial $3x^3 - 12x$ is factored completely as follows:

$$3x^3 - 12x = 3x(x - 2)(x + 2)$$

COMMON FACTORS

Factoring any polynomial begins with the removal of common factors. Notice that "removal" of a factor does not mean discarding it. To remove a factor is to insert parentheses and move the factor outside the parentheses as a common multiplier. The removal of common factors proceeds as follows:

1. Inspect the polynomial and find the factors which are common to all terms. These common factors, multiplied together, comprise the "largest common factor."
2. Mentally divide each term of the polynomial by the largest common factor and write the quotients within a set of parentheses.
3. Write the largest common factor outside the parentheses as a common multiplier.

For example, the expression $x^2y - xy^2$ contains xy as a factor of each term. Therefore, it is factored as follows:

$$x^2y - xy^2 = xy(x - y)$$

Other examples of factoring by the removal of common factors are found in the following expressions:

$$6m^4n + 3m^3n^2 - 3m^2n^3 = 3m^2n(2m^2 + mn - n^2)$$

$$-5z^2 - 15z = -5z(z + 3)$$

$$7x - 7y + 7z = 7(x - y + z)$$

In selecting common factors, always remove as many factors as possible from each term in order to factor completely. For example, x is a factor of $3ax^2 - 3ax$, so that $3ax^2 - 3ax$ is equal to $x(3ax - 3a)$. However, 3 and a are also factors. Thus the largest common factor is 3ax. When factored completely, the expression is as follows:

$$3ax^2 - 3ax = 3ax(x - 1)$$

Practice problems: Remove the common factors:

1. $y^2 - y$
2. $a^3b^2 - a^2b^2$
3. $2b^3 - 8b^2 - 6b$
4. $6mn^2 + 30m^2n$
5. $\frac{2}{3}x - \frac{1}{3}y + \frac{1}{3}$

Answers:

1. $y(y - 1)$
2. $a^2b^2(a - 1)$
3. $2b(b^2 - 4b - 3)$
4. $6mn(n + 5m)$
5. $\frac{1}{3}(2x - y + 1)$

LITERAL EXPONENTS

It is frequently necessary to remove common factors involving literal exponents; that is, exponents composed of letters rather than numbers. A typical expression involving literal exponents is $x^{2a} + x^a$, in which x^a is a common factor. The factored form is $x^a(x^a + 1)$. Another example of this type is $a^{m+n} + 2a^m$. Remember that a^{m+n} is equivalent to $a^m \cdot a^n$. Thus the factored form is as follows:

$$\begin{aligned} a^{m+n} + 2a^m &= a^m \cdot a^n + 2a^m \\ &= a^m(a^n + 2) \end{aligned}$$

BINOMIAL FORM

The distinctions between monomial, binomial, and trinomial factors are discussed in detail in chapter 9 of this course. An expression such as $a(x + y) + b(x + y)$ has a common factor in binomial form. The factor $(x + y)$ can be removed from both terms, with the following result:

$$a(x + y) + b(x + y) = (x + y)(a + b)$$

Sometimes it is easier to see this if a single letter is substituted temporarily for the binomial. Thus, let $(x + y) = n$, so that $a(x + y) + b(x + y)$ reduces to $(an + bn)$. The factored form is $n(a + b)$, which becomes $(x + y)(a + b)$ when n is replaced by its equal, $(x + y)$.

Another form of this type is $x(y - z) - w(z - y)$. Notice that this expression could be factored easily if the binomial in the second term were $(y - z)$. We can show that $-w(z - y)$ is equivalent to $+w(y - z)$, as follows:

$$\begin{aligned} -w(z - y) &= -w\left[(-1) \cdot (-1) \cdot z + (-1) \cdot y\right] \\ &= -w\left\{(-1)\left[(-1)z + y\right]\right\} \\ &= (-w)(-1)\left[-z + y\right] \\ &= +w(y - z) \end{aligned}$$

Substituting $+w(y - z)$ for $-w(z - y)$ in the original expression, we may now factor as follows:

$$\begin{aligned} x(y - z) - w(z - y) &= x(y - z) + w(y - z) \\ &= (y - z)(x + w) \end{aligned}$$

In factoring an expression such as $ax + bx + ay + by$, common monomial factors are removed first, as follows:

$$ax + bx + ay + by = x(a + b) + y(a + b)$$

Having removed the common monomial factors, we then remove the common binomial factor to obtain $(a + b)(x + y)$.

Notice that we could have rewritten the expression as $ax + ay + bx + by$, based on the commutative law of addition, which states that the sum of two or more terms is the same regardless of the order in which they are arranged. The first step in factoring would then produce $a(x + y) + b(x + y)$ and the final form would be $(x + y)(a + b)$. This is equivalent to $(a + b)(x + y)$, by the commutative law of multiplication, which states that the product of two or more factors is the same regardless of the order in which they are arranged.

Practice problems. Factor each of the following:

1. $x^{3a} + 3x^{2a}$
2. $xy^2 + y + x^2y + x$
3. $e^x + 4e^{4x}$
4. $7(x^2 + y^2) - 3z(x^2 + y^2)$
5. $a^2 + ab - ac - cb$
6. $\frac{1}{2}e^2r - \frac{1}{6}er^2$
7. $a^{x+2} + a^2$
8. $xy - 3x - 2y + 6$

Answers:

1. $x^{2a}(x^a + 3)$
2. $(xy + 1)(x + y)$
3. $e^x(1 + 4e^{3x})$
4. $(x^2 + y^2)(7 - 3z)$
5. $(a + b)(a - c)$
6. $\frac{1}{2}er(e - \frac{1}{3}r)$
7. $a^2(a^x + 1)$
8. $(y - 3)(x - 2)$

BINOMIAL FACTORS

After any common factor has been removed from a polynomial, the remaining polynomial factor must be examined further for other factors. Skill in factoring is principally the ability to recognize certain types of products such as the square of a sum or difference. Therefore, it is important to be familiar with the special products discussed in chapter 9.

DIFFERENCE OF TWO SQUARES

In chapter 9 we learned that the product of the sum and difference of two numbers is the difference of their squares. Thus, $(a + b)(a - b) = a^2 - b^2$. Conversely, if a binomial is the difference of two squares, its factors are the sum and difference of the square roots. For example, in $9a^2 - 4b^2$ both $9a^2$ and $4b^2$ are perfect squares. The square roots are 3a and 2b, respectively. Connect these square roots with a plus sign to get one factor of $9a^2 - 4b^2$ and with a minus sign to get the other factor. The two binomial factors are $3a - 2b$ and $3a + 2b$. Therefore, factored completely, the binomial can be written as follows:

$$9a^2 - 4b^2 = (3a - 2b)(3a + 2b)$$

We may check to see if these factors are correct by multiplying them together to see if their product is the original binomial.

The expression $20x^3y - 5xy^3$ reduces to the difference of two squares after the common factor 5xy is removed. Completely factored, this expression produces the following:

$$\begin{aligned} 20x^3y - 5xy^3 &= 5xy(4x^2 - y^2) \\ &= 5xy(2x - y)(2x + y) \end{aligned}$$

Other examples that show the difference of two squares in factored form are as follows:

$$\begin{aligned} 49 - 16 &= (7 + 4)(7 - 4) \\ 16a^2 - 4x^2 &= 4(4a^2 - x^2) \\ &= 4(2a + x)(2a - x) \\ 4x^2y - 9y &= y(4x^2 - 9) \\ &= y(2x + 3)(2x - 3) \end{aligned}$$

Practice problems: Factor each of the following:

1. $a^2 - b^2$	5. $x^2 - y^2$
2. $b^2 - 9$	6. $y^2 - 36$
3. $a^2b^2 - 1$	7. $1 - 4y^2$
4. $a^2 - 144$	8. $9a^2 - 16$

Answers:

1. $(a + b)(a - b)$	5. $(x + y)(x - y)$
2. $(b + 3)(b - 3)$	6. $(y + 6)(y - 6)$
3. $(ab + 1)(ab - 1)$	7. $(1 + 2y)(1 - 2y)$
4. $(a + 12)(a - 12)$	8. $(3a + 4)(3a - 4)$

SPECIAL BINOMIAL FORMS

Special cases involving binomial expressions are frequently encountered. All such expressions may be factored by reference to general formulas, but these formulas are beyond the scope of this course. For our purposes, analysis of some special cases will be sufficient.

Even Exponents

When the exponents on both terms of the binomial are even, the expression may be treated as the sum or difference of two squares. For example, $x^6 - y^6$ can be rewritten as $(x^3)^2 - (y^3)^2$ which results in the following factored form:

$$x^6 - y^6 = (x^3 - y^3)(x^3 + y^3)$$

In general, a binomial with even exponents has the form $x^{2m} \pm y^{2n}$, since all even numbers have 2 as a factor. If the connecting sign is positive, the expression may not be factorable; for example, $x^2 + y^2$, $x^4 + y^4$, and $x^8 + y^8$ are all nonfactorable binomials. If the connecting sign is negative, a binomial with even exponents is factorable as follows:

$$x^{2m} - y^{2n} = (x^m - y^n)(x^m + y^n)$$

A special case which is particularly important because it occurs so often is the binomial which has the numeral 1 as one of its terms. For example, $x^4 - 1$ is factorable as the difference of two squares, as follows:

$$\begin{aligned} x^4 - 1 &= (x^2 - 1)(x^2 + 1) \\ &= (x - 1)(x + 1)(x^2 + 1) \end{aligned}$$

Odd Exponents

Two special cases involving odd exponents are of particular importance. These are the sum of two cubes and the difference of two cubes. Examples of the sum and difference of two cubes, showing their factored forms, are as follows:

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$
$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

Notice that each of these factored forms involves a first degree binomial factor ((x + y) in the first case and (x - y) in the second). The connecting sign in the first degree binomial factor corresponds to the connecting sign in the original unfactored binomial.

We are now in a position to give the completely factored form of $x^6 - y^6$, as follows:

$$\begin{aligned} x^6 - y^6 &= (x^3 - y^3)(x^3 + y^3) \\ &= (x - y)(x^2 + xy + y^2) \\ &\qquad (x + y)(x^2 - xy + y^2) \end{aligned}$$

In general, (x + y) is a factor of $(x^n + y^n)$ if n is odd. If n is even, $(x^n + y^n)$ is not factorable unless it can be expressed as the sum of two cubes. When the connecting sign is negative, the binomial is always factorable if n is a whole number greater than 1. That is, (x - y) is a factor of $(x^n - y^n)$ for both odd and even values of n.

The special case in which one of the terms of the binomial is the numeral 1 occurs frequently. An example of this is $x^3 + 1$, which is factorable as the sum of two cubes, as follows:

$$x^3 + 1 = (x + 1)(x^2 - x + 1)$$

In a similar manner, $1 + x^6$ can be treated as the sum of two cubes and factored as follows:

$$\begin{aligned} 1 + x^6 &= 1 + (x^2)^3 \\ &= (1 + x^2)(1 - x^2 + x^4) \end{aligned}$$

Practice problems. In each of the following problems, factor completely:

1. $x^4 - y^4$	4. $x^3 - y^3$	7. $1 - x^4$
2. $m^3 + n^3$	5. $a^9 - b^9$	8. $x^6 + 1$
3. $x^6 - y^6$	6. $x^{2a} - y^{2b}$	9. $1 - x^3$

Answers:

1. $(x + y)(x - y)(x^2 + y^2)$
2. $(m + n)(m^2 - mn + n^2)$
3. $(x + y)(x - y)(x^2 + xy + y^2)(x^2 - xy + y^2)$
4. $(x - y)(x^2 + xy + y^2)$
5. $(a - b)(a^2 + ab + b^2)(a^6 + a^3b^3 + b^6)$
6. $(x^a - y^b)(x^a + y^b)$
7. $(1 + x^2)(1 - x)(1 + x)$
8. $(x^2 + 1)(x^4 - x^2 + 1)$
9. $(1 - x)(1 + x + x^2)$

TRINOMIAL SQUARES

A trinomial that is the square of a binomial is called a TRINOMIAL SQUARE. Trinomials that are perfect squares factor into either the square of a sum or the square of a difference. Recalling that $(x + y)^2 = x^2 + 2xy + y^2$ and $(x - y)^2 = x^2 - 2xy + y^2$, the form of a trinomial square is apparent. The first term and the last term are perfect squares and their signs are positive. The middle term is twice the product of the square roots of these two numbers. The sign of the middle term is plus if a sum has been squared; it is minus if a difference has been squared.

The polynomial $16x^2 - 8xy + y^2$ is a trinomial in which the first term, $16x^2$, and the last term, y^2, are perfect squares with positive signs. The square roots are 4x and y. Twice the product of these square roots is 2(4x)(y) = 8xy. The middle term is preceded by a minus sign indicating that a difference has been squared. In factored form this trinomial is as follows:

$$16x^2 - 8xy + y^2 = (4x - y)^2$$

To factor the trinomial, we simply take the square roots of the end terms and join them with a plus sign if the middle term is preceded by a plus or with a minus if the middle term is preceded by a minus.

The terms of a trinomial may appear in any order. Thus, $8xy + y^2 + 16x^2$ is a trinomial square and may be factored as follows:

$$8xy + y^2 + 16x^2 = 16x^2 + 8xy + y^2 = (4x + y)^2$$

Practice problems. Among the following expressions, factor those which are trinomial squares:

1. $y^2 - 8y + 16$
2. $16y^2 + 30x + 9$
3. $36 + 12x + x^2$
4. $a^2 + 2ab + b^2$
5. $12y + 9y^2 - 4$
6. $4x^2 + y^2 + 4xy$
7. $9 - 6cd + c^2d^2$
8. $x^4 + 4x^2 + 4$

Answers:

1. $(y - 4)^2$
2. Not a trinomial square
3. $(6 + x)^2$
4. $(a + b)^2$
5. Not a trinomial square
6. $(2x + y)^2$
7. $(3 - cd)^2$
8. $(x^2 + 2)^2$

SUPPLYING THE MISSING TERM

Skill in recognizing trinomial squares may be improved by practicing the solution of problems which require supplying a missing term. For example, the expression $y^2 + (?) + 16$ can be made to form a perfect trinomial square by supplying the correct term to fill the parentheses.

The middle term must be twice the product of the square roots of the two perfect square terms; that is, $(2)(4)(y)$, or $8y$. Check: $y^2 + 8y + 16 = (y + 4)^2$. The missing term is $8y$.

Suppose that we wish to supply the missing term in $16x^2 + 24xy + (?)$ so that the three terms will form a perfect trinomial square. The square root of the first term is $4x$. One-half the middle term is $12xy$. Divide $12xy$ by $4x$. The result is $3y$ which is the square root of the last term. Thus, our missing term is $9y^2$. Checking, we find that $(4x + 3y)^2 = 16x^2 + 24xy + 9y^2$.

Practice problems. In each of the following problems, supply the missing term to form a perfect trinomial square:

1. $x^2 + (?) + y^2$
2. $t^2 + (?) + 25$
3. $9a^2 - (?) + 25b^2$
4. $4m^2 + 16m + (?)$
5. $x^2 + 4x + (?)$
6. $c^2 - 6cd + (?)$

Answers:

1. $2xy$
2. $10t$
3. $30ab$
4. 16
5. 4
6. $9d^2$

OTHER TRINOMIALS

It is sometimes possible to factor trinomials that are not perfect squares. Following are some examples of such trinomials, and the expressions of which they are products:

1. $(x + 3)(x + 4) = x^2 + 7x + 12$
2. $(x - 3)(x - 4) = x^2 - 7x + 12$
3. $(x - 3)(x + 4) = x^2 + x - 12$
4. $(x + 3)(x - 4) = x^2 - x - 12$

It is apparent that trinomials like these may be factored into binomials as shown. Notice how the trinomial in each of the preceding examples is formed. The first term is the square of the common term of the binomial factors. The second term is the algebraic sum of their unlike terms times their common term. The third term is the product of their unlike terms.

Such trinomials may be factored as the product of two binomials if there are two numbers such that their algebraic sum is the coefficient of the middle term and their product is the last term.

For example, let us factor the expression $x^2 - 12x + 32$. If the expression is factorable, there will be a common term, x, in each of the binomial factors. We begin factoring by placing this term within each set of parentheses, as follows:

$$(x \quad\quad)(x \quad\quad)$$

Next, we must find the other terms that are to go in the parentheses. They will be two numbers such that their algebraic sum is -12 and their product is +32. We see that -8 and -4 satisfy the conditions. Thus, the following expression results:

$$x^2 - 12x + 32 = (x - 8)(x - 4)$$

It is of value in factoring to note some useful facts about trinomials. If both the second and third terms of the trinomial are positive, the signs of the terms to be found are positive as in example 1 of this section. If the second term is negative and the last is positive, both terms to be found will be negative as in example 2. If the third term of the trinomial is negative, one of the terms to be found is positive and the other is negative as in examples 3 and 4. Concerning this last case, if the second term is positive as in example 3, the positive term in the factors has the greater numerical value. If the second term is negative as in example 4, the negative term in the factors has the greater numerical value.

It should be remembered that not all trinomials are factorable. For example, $x^2 + 4x + 2$ cannot be factored since there are no two rational numbers whose product is 2 and whose sum is 4.

Practice problems. Factor completely, in the following problems:

1. $y^2 + 15y + 50$	5. $x^2 - 12x - 45$
2. $y^2 - 2y - 24$	6. $x^2 - 15x + 56$
3. $x^2 + 8x - 48$	7. $x^2 + 2x - 48$
4. $x^2 - 4x - 60$	8. $x^2 + 14x + 24$

Answers:

1. $(y + 5)(y + 10)$	5. $(x - 15)(x + 3)$
2. $(y - 6)(y + 4)$	6. $(x - 7)(x - 8)$
3. $(x + 12)(x - 4)$	7. $(x - 6)(x + 8)$
4. $(x - 10)(x + 6)$	8. $(x + 12)(x + 2)$

Thus far we have considered only those expressions in which the coefficient of the first term is 1. When the coefficient of the first term is other than 1, the expression can be factored as shown in the following example:

$$6x^2 - x - 2 = (2x + 1)(3x - 2)$$

Although this result can be obtained by the trial and error method, the following procedure saves time and effort. First, find two numbers whose sum is the coefficient of the second term (-1 in this example) and whose product is equal to the product of the third term and the coefficient of the first term (in this example, (6)(-2) or -12). By inspection, the desired numbers are found to be -4 and +3. Using these two numbers as coefficients for x, we can rewrite the original expression as $6x^2 - 4x + 3x - 2$ and factor as follows:

$$\begin{aligned} 6x^2 - 4x + 3x - 2 &= 2x(3x - 2) + 1(3x - 2) \\ &= (2x + 1)(3x - 2) \end{aligned}$$

Practice problems. Factor completely, in the following problems:

1. $2x^2 + 13x + 21$	3. $15x^2 - 16x - 7$
2. $16x^2 + 26x + 3$	4. $12x^2 - 8x - 15$

Answers:

1. $(2x + 7)(x + 3)$	3. $(3x + 1)(5x - 7)$
2. $(2x + 3)(8x + 1)$	4. $(6x + 5)(2x - 3)$

REDUCING FRACTIONS TO LOWEST TERMS

There are many useful applications of factoring. One of the most important is that of simplifying algebraic fractions. Fractions that contain algebraic expressions in the numerator or denominator, or both, can be reduced to lower terms, if there are factors common to numerator and denominator. If the terms of a fraction are monomials, common factors are immediately apparent, as in the following expression:

$$\frac{3x^2y}{6xy} = \frac{3xy(x)}{3xy(2)} = \frac{x}{2}$$

If the terms of a fraction are polynomials, the polynomials must be factored in order to recognize the existence of common factors, as in the following two examples:

1. $$\frac{a - b}{a^2 - 2ab + b^2} = \frac{a - b}{(a - b)(a - b)} = \frac{1}{(a - b)}$$

2. $$\frac{4x^2 - 9}{6x^2 - 9x} = \frac{(2x + 3)(2x - 3)}{3x(2x - 3)} = \frac{(2x + 3)}{3x}$$

Notice that without the valuable process of factoring, we would be forced to use the fractions in their more complicated form. When there are factors common to both numerator and denominator, it is obviously more practical to cancel them (first using the factoring process) before proceeding.

Practice problems. Reduce to lowest terms in each of the following:

1. $\frac{12}{6x + 12}$	4. $\frac{y^2 - 25}{y^2 - 8y + 15}$
2. $\frac{a^2 - b^2}{a^2 - 2ab + b^2}$	5. $\frac{a^2 - 5a - 24}{a^2 - 64}$
3. $\frac{y^2 - 14y + 45}{y^2 - 8y - 9}$	6. $\frac{4x^2y - 9y}{4x^2 + 12x + 9}$

Answers:

1. $\frac{2}{x + 2}$
2. $\frac{a + b}{a - b}$
3. $\frac{y - 5}{y + 1}$
4. $\frac{y + 5}{y - 3}$
5. $\frac{a + 3}{a + 8}$
6. $\frac{y(2x - 3)}{2x + 3}$

OPERATIONS INVOLVING FRACTIONS

Addition, subtraction, multiplication, and division operations involving algebraic fractions are often simplified by means of factoring, whereas they would be quite complicated without the use of factoring.

MULTIPLYING FRACTIONS

Multiplication of fractions that contain polynomials is similar to multiplication of fractions that contain only arithmetic numbers. If this fact is kept in mind, the student will have little difficulty in mastering multiplication in algebra. For instance, we recall that to multiply a fraction by a whole number, we simply multiply the numerator by the whole number. This is illustrated in the following example:

Arithmetic: $4 \times \frac{3}{17} = \frac{12}{17}$

Algebra: $(x - 4) \cdot \frac{3}{x^2 - 5} = \frac{3x - 12}{x^2 - 5}$

Sometimes the work may be simplified by factoring and canceling before carrying out the multiplication. The following example illustrates this:

$$(2a - 8) \cdot \frac{3}{a^2 - 8a + 16} = \frac{2\cancel{(a - 4)}}{1} \cdot \frac{3}{(a - 4)\cancel{(a - 4)}}$$

$$= \frac{2(3)}{a - 4} = \frac{6}{a - 4}$$

When the multiplier is a fraction, the rules of arithmetic remain applicable—that is, multiply numerators together and denominators together. This is illustrated as follows:

Arithmetic: $\frac{4}{5} \times \frac{2}{3} = \frac{8}{15}$

Algebra: $\frac{a + b}{a - b} \cdot \frac{a}{a - b} = \frac{a(a + b)}{(a - b)^2}$

Where possible, the work may be considerably reduced by factoring, canceling, and then carrying out the multiplication, as in the following example:

$$\frac{x^2 - 2x + 1}{x^2 - 9} \cdot \frac{x^2 + x - 6}{x^2 - 1}$$

$$= \frac{(x - 1)\cancel{(x - 1)}}{\cancel{(x + 3)}(x - 3)} \cdot \frac{\cancel{(x + 3)}(x - 2)}{(x + 1)\cancel{(x - 1)}}$$

$$= \frac{(x - 1)(x - 2)}{(x - 3)(x + 1)} = \frac{x^2 - 3x + 2}{x^2 - 2x - 3}$$

Although the factors may be multiplied to form two trinomials as shown, it is usually sufficient to leave the answer in factored form.

Practice problems. In the following problems, multiply as indicated:

1. $5a^2 \cdot \frac{3b}{a + b}$
2. $\frac{x + y}{x^2} \cdot \frac{x - y}{x - 1}$
3. $\frac{a^2 + 2ab + b^2}{a^2 - b^2} \cdot \frac{6a}{3a + 3b}$
4. $\frac{a - 1}{2a^2 + 4a + 2} \cdot \frac{(a + 1)^2}{a - 1}$

Answers:

1. $\frac{15a^2b}{a + b}$
2. $\frac{x^2 - y^2}{x^3 - x^2}$
3. $\frac{2a}{(a - b)}$
4. $\frac{1}{2}$

DIVIDING FRACTIONS

The rules of arithmetic apply to the division of algebraic fractions; as in arithmetic, simply invert the divisor and multiply, as follows:

Arithmetic: $\frac{3}{8} \div \frac{9}{16} = \frac{3}{8} \times \frac{16}{9}$

$$= \frac{\cancel{3}}{\cancel{8}} \times \frac{\cancel{(8)}(2)}{\cancel{(3)}(3)} = \frac{2}{3}$$

Algebra: $\frac{x - 3y}{x + 3y} \div \frac{x^2 - 6xy + 9y^2}{x^2 + 7xy + 12y^2}$

$$= \frac{x - 3y}{x + 3y} \cdot \frac{x^2 + 7xy + 12y^2}{x^2 - 6xy + 9y^2}$$

$$= \frac{\cancel{x - 3y}}{\cancel{x + 3y}} \cdot \frac{\cancel{(x + 3y)}(x + 4y)}{\cancel{(x - 3y)}(x - 3y)}$$

$$= \frac{x + 4y}{x - 3y}$$

Practice problems. In the following problems, divide and reduce to lowest terms:

1. $\frac{x - 2}{x^2 + 4x + 4} \div \frac{1}{x^2 - 4}$

2. $\frac{2a - 1}{a^3 + 3a} \div \frac{a + 1}{a^2 + 3}$

3. $\frac{a^3 - 4a^2 + 3a}{a + 2} \div (a - 3)$

4. $\frac{6t + 12}{9t^2 + 6t - 24} \div \frac{8t - 12}{15t - 20}$

Answers:

1. $\frac{(x - 2)^2}{x + 2}$
2. $\frac{2a - 1}{a^2 + a}$
3. $\frac{a(a - 1)}{a + 2}$
4. $\frac{5}{4t - 6}$

ADDING AND SUBTRACTING FRACTIONS

The rules of arithmetic for adding and subtracting fractions are applicable to algebraic fractions. Fractions that are to be combined by addition or subtraction must have the same denominator. The numerators are then combined according to the operation indicated and the result is placed over the denominator. For example, in the expression

$$\frac{x - 4}{x - 2} + \frac{2 - 11x}{2 - x}$$

the second denominator will be the same as the first, if its sign is changed. The value of the fraction will remain the same if the sign of the numerator is also changed. Thus, we have the following simplification:

$$\frac{x - 4}{x - 2} + \frac{2 - 11x}{2 - x} = \frac{x - 4}{x - 2} + \frac{-(2 - 11x)}{-(2 - x)}$$

$$= \frac{x - 4}{x - 2} + \frac{11x - 2}{x - 2}$$

$$= \frac{x - 4 + 11x - 2}{x - 2}$$

$$= \frac{12x - 6}{x - 2}$$

$$= \frac{6(2x - 1)}{x - 2}$$

When the denominators are not the same, we must reduce all fractions to be added or subtracted to a common denominator and then proceed.

Consider, for example,

$$\frac{4}{x^2 - 4} + \frac{3}{x^2 - 4x - 12}$$

We first must find the least common denominator (LCD). Remember this is the least number that is exactly divisible by each of the denominators. To find such a number, as in arithmetic, we first separate each of the denominators into prime factors. The LCD will contain all of the various prime factors, each one as many times as it occurs in any of the denominators.

Factoring, we have

$$\frac{4}{(x + 2)(x - 2)} + \frac{3}{(x - 6)(x + 2)}$$

and the LCD is $(x + 2)(x - 2)(x - 6)$. Rewriting the fractions with this denominator and adding numerators, we have the following expression:

$$\frac{4(x - 6)}{(x + 2)(x - 2)(x - 6)} + \frac{3(x - 2)}{(x + 2)(x - 2)(x - 6)}$$

$$= \frac{4(x - 6) + 3(x - 2)}{\text{LCD}}$$

$$= \frac{4x - 24 + 3x - 6}{\text{LCD}}$$

$$= \frac{7x - 30}{(x + 2)(x - 2)(x - 6)}$$

As another example, consider

$$\frac{4}{x + 3} - \frac{x + 2}{x^2 + 4x + 3}$$

Factoring the denominator of the second fraction, we find that the LCD is $(x + 3)(x + 1)$. Rewriting the original fractions with the LCD as denominator, we may now combine the fractions as follows:

$$\frac{4(x + 1)}{(x + 3)(x + 1)} - \frac{(x + 2)}{(x + 3)(x + 1)}$$

$$= \frac{4x + 4 - x - 2}{(x + 3)(x + 1)}$$

$$= \frac{3x + 2}{(x + 3)(x + 1)}$$

Practice problems. Perform the indicated operations in each of the following problems:

1. $\frac{3x - 4}{x^2 + x - 2} - \frac{x - 2}{x - 1}$

2. $\frac{3a}{a^2 - 9} - \frac{3}{3 - a}$

3. $\frac{x - 3}{3x} + \frac{x + 2}{2x}$

4. $\frac{1}{a^4 - 1} - \frac{1}{a + 1}$

5. $\frac{3}{(a + 4)^2} - \frac{2}{a(a + 4)} + \frac{1}{6(a + 4)}$

Answers:

1. $\frac{3x - x^2}{(x + 2)(x - 1)}$

2. $\frac{6a + 9}{(a + 3)(a - 3)}$

3. $\frac{5}{6}$

4. $\frac{2 - a^3 + a^2 - a}{(a^2 + 1)(a + 1)(a - 1)}$

5. $\frac{a^2 + 10a - 48}{6a(a + 4)^2}$